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Exact Solutions of Some Fractional Power Series

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Abstract: **In this paper, we obtain the exact solutions of two fractional power series. A new multiplication of fractional power series and Jumarie type of Riemann-Liouville (R-L) fractional calculus play important roles in this article. In fact, our results are generalizations of ordinary calculus results.**

Keyword: **Exact solutions, fractional power series, new multiplication, Jumarie type of R-L fractional calculus.**

I. INTRODUCTION

During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, and some others who reported interesting results within fractional calculus. In recent years, fractional calculus has become an increasingly popular research area due to its effective applications in different scientific fields such as economics, engineering, dynamics, biology, control theory, mechanics, chaos theory, and so on [1-10].

However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-14]. Because Jumarie's modified R-L fractional derivative helps avoid non-zero fractional derivative of constant functions, it is easier to use this definition to associate fractional calculus with classical calculus.

In this paper, we find the exact solutions of the following two α-fractional power series:

$$\sum_{n=1}^{\infty} \frac{(4n)!}{\Gamma((4n+1)\alpha+1)} x^{(4n+1)\alpha} , \tag{1}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(4n\alpha+1)} x^{4n\alpha} . \tag{2}$$

where $0 < \alpha \leq 1$. Jumarie type of R-L fractional calculus and a new multiplication of fractional power series play important roles in this paper. And our results are generalizations of the results in classical calculus.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([15])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x_0}^{x} \frac{f(t)-f(x_0)}{(x-t)^{\alpha}} dt . \tag{3}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} dt , \tag{4}$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer p, we define $\left({}_{x_0}D_x^{\alpha}\right)^p[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the p-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([16])**:** *If α, β, x_0, C are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{0}D_x^{\alpha}\right)\left[x^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{5}$$

and

$$\left({}_{0}D_x^{\alpha}\right)[C] = 0. \tag{6}$$

Next, we introduce the definition of fractional power series.

Definition 2.3: Suppose that x and a_n are real numbers for all n, and $0 < \alpha \leq 1$. If the function $f_{\alpha}: [a, b] \to R$ can be expressed as $f_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} x^{n\alpha}$, then we say that $f_{\alpha}(x^{\alpha})$ is a α-fractional power series.

In the following, a new multiplication of fractional power series is introduced.

Definition 2.4 ([17])**:** If $0 < \alpha \leq 1$. If $f_{\alpha}(x^{\alpha})$ and $g_{\alpha}(x^{\alpha})$ are two α-fractional power series,

$$f_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} x^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n}, \tag{7}$$

$$g_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} x^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n}. \tag{8}$$

Then we define

$$\begin{aligned} & f_{\alpha}(x^{\alpha}) \otimes g_{\alpha}(x^{\alpha}) \\ & = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} x^{n\alpha} \otimes \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} x^{n\alpha} \\ & = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right) x^{n\alpha}. \end{aligned} \tag{9}$$

Equivalently,

$$\begin{aligned} & f_{\alpha}(x^{\alpha}) \otimes g_{\alpha}(x^{\alpha}) \\ & = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n} \otimes \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n} \\ & = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m\right) \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n}. \end{aligned} \tag{10}$$

Definition 2.5 ([18])**:** Assume that $0 < \alpha \leq 1$, and $f_{\alpha}(x^{\alpha})$, $g_{\alpha}(x^{\alpha})$ are α-fractional power series,

$$f_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} x^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n}, \tag{11}$$

$$g_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} x^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n}. \tag{12}$$

The compositions of $f_{\alpha}(x^{\alpha})$ and $g_{\alpha}(x^{\alpha})$ are defined by

$$(f_{\alpha} \circ g_{\alpha})(x^{\alpha}) = f_{\alpha}\left(g_{\alpha}(x^{\alpha})\right) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(g_{\alpha}(x^{\alpha})\right)^{\otimes n}, \tag{13}$$

and

$$(g_{\alpha} \circ f_{\alpha})(x^{\alpha}) = g_{\alpha}\left(f_{\alpha}(x^{\alpha})\right) = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(f_{\alpha}(x^{\alpha})\right)^{\otimes n}. \tag{14}$$

Definition 2.6 ([19])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_{\alpha}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha}\right)^{\otimes n}. \tag{15}$$

And the α-fractional logarithmic function $Ln_{\alpha}(x^{\alpha})$ is the inverse function of $E_{\alpha}(x^{\alpha})$.

Definition 2.7 ([20])**:** The α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n\alpha+1)} x^{2n\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2n}, \quad (16)$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma((2n+1)\alpha+1)} x^{(2n+1)\alpha} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(2n+1)}. \quad (17)$$

III. EXAMPLES

In this section, we find the exact solutions of two fractional power series.

Example 3.1: Suppose that $0 < \alpha \leq 1$. Find the α-fractional power series

$$\sum_{n=1}^{\infty} \frac{(4n)!}{\Gamma((4n+1)\alpha+1)} x^{(4n+1)\alpha}, \quad (18)$$

where $-1 < \frac{1}{\Gamma(\alpha+1)} x^\alpha < 1$, and $\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4} < 1$.

Solution $\sum_{n=1}^{\infty} \frac{(4n)!}{\Gamma((4n+1)\alpha+1)} x^{(4n+1)\alpha}$

$$= \sum_{n=1}^{\infty} \frac{1}{4n+1} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(4n+1)}$$

$$= \sum_{n=1}^{\infty} \left({}_0I_x^\alpha\right) \left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4n}\right]$$

$$= \left({}_0I_x^\alpha\right) \left[\sum_{n=1}^{\infty} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4n}\right]$$

$$= \left({}_0I_x^\alpha\right) \left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4} \otimes \sum_{n=0}^{\infty} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4n}\right]$$

$$= \left({}_0I_x^\alpha\right) \left[\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4} \otimes \left[1 - \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4}\right]^{\otimes -1}\right]$$

$$= \left({}_0I_x^\alpha\right) \left[-1 + \left[1 - \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 4}\right]^{\otimes -1}\right]$$

$$= \left({}_0I_x^\alpha\right) \left[-1 + \frac{1}{2}\left[1 + \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2}\right]^{\otimes -1} + \frac{1}{2}\left[1 - \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2}\right]^{\otimes -1}\right]$$

$$= \frac{1}{2}\left({}_0I_x^\alpha\right) \left[\left[1 - \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2}\right]^{\otimes -1}\right] + \frac{1}{2}\left({}_0I_x^\alpha\right) \left[\left[1 + \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2}\right]^{\otimes -1}\right] - \left({}_0I_x^\alpha\right)[1]$$

$$= \frac{1}{4} Ln_\alpha\left(\left(1 + \frac{1}{\Gamma(\alpha+1)} x^\alpha\right) \otimes \left(1 - \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1}\right) + \frac{1}{2} arctan_\alpha(x^\alpha) - \frac{1}{\Gamma(\alpha+1)} x^\alpha. \quad (19)$$

Example 3.2: Let $0 < \alpha \leq 1$. Find the α-fractional power series

$$\sum_{n=0}^{\infty} \frac{1}{\Gamma(4n\alpha+1)} x^{4n\alpha}. \quad (20)$$

Solution Let $y_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(4n\alpha+1)} x^{4n\alpha}$, then the α-fractional derivatives of $y_\alpha(x^\alpha)$

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = \sum_{n=1}^{\infty} \frac{1}{\Gamma((4n-1)\alpha+1)} x^{(4n-1)\alpha}, \quad (21)$$

$$\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)] = \sum_{n=1}^{\infty} \frac{1}{\Gamma((4n-2)\alpha+1)} x^{(4n-2)\alpha}, \quad (22)$$

$$\left({}_0D_x^\alpha\right)^3[y_\alpha(x^\alpha)] = \sum_{n=1}^{\infty} \frac{1}{\Gamma((4n-3)\alpha+1)} x^{(4n-3)\alpha}, \quad (23)$$

$$\left({}_{0}D_{x}^{\alpha}\right)^{4}[y_{\alpha}(x^{\alpha})]=\sum_{n=1}^{\infty}\frac{1}{\Gamma((4n-4)\alpha+1)}x^{(4n-4)\alpha}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(4n\alpha+1)}x^{4n\alpha}. \quad (24)$$

Therefore,

$$\left({}_{0}D_{x}^{\alpha}\right)^{4}[y_{\alpha}(x^{\alpha})]-y_{\alpha}(x^{\alpha})=0, \quad (25)$$

and

$$y_{\alpha}(0)=1,\left({}_{0}D_{x}^{\alpha}\right)[y_{\alpha}(x^{\alpha})](0)=\left({}_{0}D_{x}^{\alpha}\right)^{2}[y_{\alpha}(x^{\alpha})](0)=\left({}_{0}D_{x}^{\alpha}\right)^{3}[y_{\alpha}(x^{\alpha})](0)=0. \quad (26)$$

This is a 4-th order linear α-fractional differential equation, and we can obtain the general solution is

$$y_{\alpha}(x^{\alpha})=C_{1}E_{\alpha}(x^{\alpha})+C_{2}E_{\alpha}(-x^{\alpha})+C_{3}cos_{\alpha}(x^{\alpha})+C_{4}sin_{\alpha}(x^{\alpha}), \quad (27)$$

where C_1, C_2, C_3, C_4 are constants. Using initial value conditions yields

$$C_{1}=C_{2}=\frac{1}{4}, C_{3}=\frac{1}{2}, \text{ and } C_{4}=0. \quad (28)$$

Thus,

$$y_{\alpha}(x^{\alpha})=\frac{1}{4}E_{\alpha}(x^{\alpha})+\frac{1}{4}E_{\alpha}(-x^{\alpha})+\frac{1}{2}cos_{\alpha}(x^{\alpha}). \quad (29)$$

That is,

$$\sum_{n=0}^{\infty}\frac{1}{\Gamma(4n\alpha+1)}x^{4n\alpha}=\frac{1}{4}E_{\alpha}(x^{\alpha})+\frac{1}{4}E_{\alpha}(-x^{\alpha})+\frac{1}{2}cos_{\alpha}(x^{\alpha}). \quad (30)$$

IV. CONCLUSION

In this paper, we find the exact solutions of two fractional power series. Jumarie's modified R-L fractional calculus and a new multiplication of fractional power series play important roles in this article. In fact, our results are generalizations of the results in traditional calculus. In the future, we will expand our research fields to engineering mathematics and fractional differential equations.

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